① Let $\hat{G}$ denote all irreducible representations of G

By what we have proved $\{\chi_\rho,\ \rho \in \hat{G}\}$ form part of an orthonormal basis of $L^2(G)$

$\Rightarrow$ We may write them as $\{\chi_{\phi_i}\}_{i=1}^{\infty}$.

Now, $\forall \rho$ $\boxed{\rho = \sum m(\rho, \phi_i)\, \phi_i}$ $m(\rho, \phi_i)$ is the multiplicity.

$$\Rightarrow \underbrace{\chi_\rho = \sum m(\rho, \phi_i)\, \chi_{\phi_i}}$$

$$\int \chi_\rho \cdot \overline{\chi_{\phi_j}} = \underline{m(\rho, \phi_j)}$$

If ρ & ψ are two representations

$$\chi_\rho = \chi_\psi \Rightarrow m(\rho, \phi_j) = m(\psi, \phi_j) \quad \forall j$$

$$\Rightarrow \rho = \psi.$$

$\chi_\rho = \sum m(\rho, \phi_i)\, \chi_{\phi_i}$

Moreover if $\int \chi_\rho \overline{\chi_\rho} = 1$

$\Rightarrow$ $\sum m^2(\rho, \phi_i) = 1$ $\Rightarrow$ only one of them say $m(\rho, \phi_{i_0}) = 1$

& $\rho = \phi_{i_0}$ $\Rightarrow$ ρ is irreducible.

This completes the proof of Thm 3 of last lecture.

Thm 2 fails if G is Not compact.

e.g.: $G = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \right\}$.

② Quaternions:

Cross product

$\to v_1 \times (v_2 \times v_3) = \langle v_1, v_3\rangle v_2 - \langle v_1, v_2\rangle v_3$

This can be checked by brutal force computation

$$\Rightarrow \quad v_1 \times (v_2 \times v_3) = \langle \cancel{v_1, v_3}\rangle v_2 - \langle v_1, v_2\rangle v_3$$
$$+ v_2 \times (v_3 \times v_1) + \langle v_2, v_1\rangle v_3 - \langle v_2, v_3\rangle v_1 = 0$$
$$+ v_3 \times (v_1 \times v_2) + \langle v_3, v_2\rangle v_1 - \langle \cancel{v_3}, v_1\rangle v_2$$

Namely $(\mathbb{R}^3, \times)$ is a Lie algebra.

$q = a + bj$ $\quad \bar{q} = \bar{a} - j\bar{b}$

$ij = k$, $jk = i$, $ki = j$

$q = x + iy + jz + kw$, $\quad \bar{q} \doteq x - iy - jz - kw$
$\quad \underbrace{x + iy}_{} + \underbrace{(z + iw)}_{} j \quad = \underbrace{(x - iy)}_{} - j\underbrace{(z - iw)}_{}$

$q(t) = x(t) + iy(t) + jz(t) + kw(t)$.

$$\begin{vmatrix} i, & j, & k \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = \begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix} i + \cdots$$

Consider unit quaternions $\quad x^2 + y^2 + z^2 + w^2 = 1$.

$jz = j(x+iy) = jx - ky = (x - iy)j = \bar{z}j$

$$\underbrace{q \cdot \bar{q}}_{} = (z_1 + z_2 j)(\bar{z}_1 - j\bar{z}_2)$$
$$= z_1\bar{z}_1 - z_1 z_2 j + z_2 z_1 j + z_2\bar{z}_2$$
$$= \underbrace{|z_1|^2 + |z_2|^2}_{}$$

$q_1 \cdot \bar{q}_2 + q_2 \cdot \bar{q}_1$

$\qquad \|$

$$\begin{cases} (z_1 + z_2 j)(\bar{w}_1 - j\bar{w}_2) = z_1\bar{w}_1 - z_1 w_2 j + z_2 w_1 j + z_2\bar{w}_2 \\ + (w_1 + w_2 j)(\bar{z}_1 - j\bar{z}_2) \quad + w_1\bar{z}_1 - z_2 w_1 j + w_2 z_1 j + w_2\bar{z}_2 \end{cases}$$

$$= \boxed{z_1\bar{w}_1 + w_1\bar{z}_1 + z_2\bar{w}_2 + w_2\bar{z}_2}$$
$$= (a_1 + ib_1)(\tilde{a}_1 - i\tilde{b}_1) + (\tilde{a}_1 + i\tilde{b}_1)(a_1 - ib_1)$$
$$+ (c_1 + id_1)(\tilde{c}_1 - i\tilde{d}_1) + (\tilde{c}_1 + i\tilde{d}_1)(c_1 - id_1)$$
$$= 2(a_1\tilde{a}_1 + b_1\tilde{b}_1 + c_1\tilde{c}_1 + d_1\tilde{d}_1)$$

Namely $\boxed{q_1 \cdot \bar{q}_2 + q_2 \cdot \bar{q}_1} = 2\underbrace{\langle R(q_1), R(q_2)\rangle}_{}$

$$R(q_1) \doteq \begin{pmatrix} a_1 \\ b_1 \\ c_1 \\ d_1 \end{pmatrix} \qquad q_1 = a_1 + ib_1 + (c_1 + d_1 i) j$$

$a^2 + b^2 + c^2 + d^2 = 1$

$|q|^2 = 1 \iff$

$(q_1 \cdot q_2)(\overline{q_1 \cdot q_2})$
$= q_1 q_2 \bar{q}_2 \bar{q}_1 = q_1\bar{q}_1 = 1$

$q = a + ib + (c + di)j = a + ib + jc + dk$

$a, b, c, d \in \mathbb{R}$

Now $\quad 0 = \overline{q'(t)}\cdot \overline{q(t)} + q(t)\cdot \overline{q'(t)}$ $\qquad \underline{q(t)\cdot\overline{q(t)} = 1}$ $\qquad \underline{|q(t)|^2 = 1}$

$\underbrace{q(0) = 1}$ $= \begin{pmatrix}1\\0\\0\\0\end{pmatrix}$

$\Rightarrow \quad q'(t)\Big|_{t=0} = \underbrace{y'(0)i + z'(0)j + w'(0)k}.$ $\qquad$ namely $x'(0) = 0$

$\Rightarrow$ $\underline{q}$ near 1 can be written as

$\underbrace{1 + q' + o(2)}$ $\qquad q' = ai + bj + ck$

Now we use x - for q

$x\cdot y - y\cdot x$ for x, y near 1.

Recall, $\begin{cases} \psi(x,y) = x^\alpha + y^\alpha + b^\alpha_{\beta\gamma} x^\beta y^\gamma + \text{higher} \\ \psi(y,x) = y^\alpha + x^\alpha + b^\alpha_{\beta\gamma} y^\beta x^\gamma \end{cases}$

$\Rightarrow \quad \psi(x,y) - \psi(y,x) = C^\alpha_{\beta\gamma} x^\beta y^\gamma + \cdots$

To our case

$$\overset{x(t)}{\underbrace{(1 + x' + o(2))}}\overset{y(t)}{\underbrace{(1 + y' + o(2))}} - \overset{y(t)}{(1 + y' + O(2))}\overset{x(t)}{(1 + x' + O(2))}$$

$= 1 + y' + x' + x'\cdot y' + o(2) - (1 + y' + x' + y'\cdot x' + O(2))$

$= \underbrace{x'\cdot y' - y'\cdot x' + o(2)}$

$= \underbrace{(a_1 i + b_1 j + c_1 k)}\cdot\underbrace{(a_2 i + b_2 j + c_2 k)} - (a_2 i + b_2 j + c_2 k)\cdot(a_1 i + b_1 j + c_1 k) + o(2)$

$= -(\underbrace{a_1 a_2 + b_1 b_2} + c_1 c_2) + (a_1 b_2 - b_1 a_2)k + (c_1 a_2 - a_1 c_2)j + (-c_1 b_2 + c_2 b_1)i$ $\qquad \begin{vmatrix} i & j & k \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}$

$- [(a_1 a_2 + b_1 b_2 + c_1 c_2) + \text{Swap}]$

$= 2 \begin{pmatrix}a_1\\b_1\\c_1\end{pmatrix} \times \begin{pmatrix}a_2\\b_2\\c_2\end{pmatrix}.$

If we renormalize $\underline{q}'$, we get the $(\mathbb{R}^3, \times)$ as the Lie algebra

Namely $S^3(1) \doteq \{ \underline{q} \mid |\underline{q}| = 1 \}$ has $(\mathbb{R}^3, \times)$ as its Lie algebra.

$$\underline{q}_1 \cdot \underline{q}_2 = (z_1 + w_1 j)(z_2 + w_2 j) = z_1 z_2 + z_1 w_2 j + w_1 \bar{z}_2 j - w_1 \bar{w}_2$$

$$\bar{\underline{q}}_2 \cdot \bar{\underline{q}}_1 = (\bar{z}_2 - j\bar{w}_2)(\bar{z}_1 - j\bar{w}_1) = \bar{z}_2\bar{z}_1 - j z_2 \bar{w}_1 - j \bar{w}_2 \bar{z}_1 - w_2 \bar{w}_1$$

$$\Rightarrow \overline{\underline{q}_1 \cdot \underline{q}_2} = \bar{\underline{q}}_2 \cdot \bar{\underline{q}}_1$$

Hence if $|\underline{q}_1|^2 = |\underline{q}_2|^2 = 1 \Rightarrow (\underline{q}_1 \cdot \underline{q}_2)(\bar{\underline{q}}_2 \cdot \bar{\underline{q}}_1) = 1.$

Namely S^3 is a Lie group.

We shall study its representation!

(1st interesting one & nontrivial. $n=1$, $=2$ Connected Lie groups can be decided.)

It holds the key for the high dimensional study as well.

③ $SU(2) \simeq S^3 \subset \mathbb{R}^4 = \mathbb{C}^2$

$\underline{q} = w_1 + w_2 j \in \mathbb{H}$ (quaternions)

For $\underline{q}' = z_1 + j z_2$ we view it as $\mathbb{C}^2$

$(z_1 + j z_2)\lambda$ $\quad$ $z_1\lambda + j z_2 \lambda$ $\quad$ $\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}\lambda$ $\quad$ $\begin{pmatrix} z_1\lambda \\ z_2\lambda \end{pmatrix}$

$\}$ $\lambda \in \mathbb{C}$ acts from the right

$C(\underline{q})$ will be the matrix representation of $\underline{q}$ acts from left

$$\underline{q} \; \underline{(w_1 + w_2 j)} \; \underbrace{(z_1 + j z_2)}_{\underline{q}'} = w_1 z_1 + w_1 j z_2 + w_2 j z_1 - w_2 z_2$$

$$= \underbrace{w_1 z_1 - w_2 z_2} + j\underbrace{(\bar{w}_1 z_2 + \bar{w}_2 z_1)}$$

$$C(\underline{q}) \cdot \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} w_1 z_1 - w_2 z_2 \\ \bar{w}_1 z_2 + \bar{w}_2 z_1 \end{pmatrix}$$

$$C(q) = \begin{pmatrix} w_1 & -w_2 \\ \overline{w_2} & \overline{w_1} \end{pmatrix} \qquad q \in S^3$$

$$\det(C(q)) = 1 \quad \& \quad C(q)\cdot \overline{C(q)^t} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$c(q_1 \cdot q_2) = \begin{pmatrix} w_1\tilde{w}_1 - w_2\overline{\tilde{w}_2} & -(w_1\tilde{w}_2 + w_2\overline{\tilde{w}_1}) \\ \overline{w_1\tilde{w}_2 + w_2\overline{\tilde{w}_1}} & \overline{w_1\tilde{w}_1 - w_2\overline{\tilde{w}_2}} \end{pmatrix}$$

$$\Uparrow$$

$$(w_1 + w_2 j)\cdot(\tilde{w}_1 + \tilde{w}_2 j)$$

$$= w_1\tilde{w}_1 + w_1\tilde{w}_2 j + w_2\overline{\tilde{w}_1} j - w_2\overline{\tilde{w}_2}$$

$$C(q_1)\cdot C(q_2)$$

$$\begin{pmatrix} w_1 & -w_2 \\ \overline{w_2} & \overline{w_1} \end{pmatrix}\begin{pmatrix} \tilde{w}_1 & -\tilde{w}_2 \\ \overline{\tilde{w}_2} & \overline{\tilde{w}_1} \end{pmatrix}$$

$$= \begin{pmatrix} w_1\tilde{w}_1 - w_2\overline{\tilde{w}_2}, & -(w_1\tilde{w}_2 + w_2\overline{\tilde{w}_1}) \\ \overline{w_2}\tilde{w}_1 + \overline{w_1}\,\overline{\tilde{w}_2}, & -\overline{w_2}\tilde{w}_2 + \overline{w_1}\,\overline{\tilde{w}_1} \end{pmatrix}$$

Namely $c(q_1 \cdot q_2) = c(q_1)\cdot c(q_2)$

$$c : S^3 := \{ q \in \mathbb{Q} \mid |q| = 1 \} \longrightarrow SU(2)$$

is a group homomorphism.

Since $\forall\ A = \begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix} \in SU(2)$, it can be checked $z_3 = -\bar{z}_2 \quad z_4 = \bar{z}_1$

$\Rightarrow \quad A = C(q)$ with $q = z_1 + z_2 j$

$\Rightarrow \quad C$ is an isomorphism.

Let $V_k :=$ degree k- polynomial of z_1, z_2

$$\mathrm{Span}\{\underbrace{z_1^k,\ z_1^{k-1} z_2,\ \dots\ z_1 z_2^{k-1},\ z_2^k}\}$$

$\dim V_k = k+1$.

$S^3 = \{ q \mid \underset{|q|=1}{q = a + bj} \}$ acts on V_k via

$C(q)$

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \to \begin{pmatrix} a & -b \\ \bar{b} & \bar{a} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} z_1' \\ z_2' \end{pmatrix}$$

Theorem 1: Let ϕ_k be the induced representation of S^3 on V_k. Then $\{\phi_k\}$ are all the irreducible representations of $S^3 \simeq SU(2)$.

(4) Proof.

The character theory reduces the study of the representation to the study of character functions.

e.g. $G = S^1$ $\qquad e^{i\theta}$

$\phi_k : G \to GL(1, \mathbb{C})$ $\qquad e^{i\theta} \to (z \to \underbrace{\boxed{e^{ik\theta}}}\, z)$

$\underbrace{\chi_{\phi_k} = e^{ik\theta}}$ $\qquad \frac{1}{2\pi}\int_{S^1} |\chi_{\phi_k}|^2 d\theta = 1.$

$\frac{1}{2\pi}\int_0^{2\pi} 1\, d\theta = 1$

$\underbrace{d\mu = \frac{1}{2\pi} d\theta}.$

& $\int \underbrace{\chi_{\phi_k} \overline{\chi_{\phi_\ell}}} = 0$ $\qquad$ Moreover $\underline{\{\chi_{\phi_k}\}_{k\in\mathbb{Z}}}$ is a complete basis in $L^2(S^1)$

$\Rightarrow$ Any $\chi_\phi = \sum c_k \chi_{\phi_k} \Rightarrow$ We obtained all representations.

We use similar idea in the proof of Theorem 1.

Remark: $\qquad \phi \otimes \psi$. $\qquad \phi : G \to GL(r, V)$

$\psi : G \to GL(r', W)$

$V \otimes W$ generated by $e_i \otimes E_j$ $\quad \{e_i\}\ \{E_j\}$

$\underbrace{}_{\text{basis of } V\otimes W}$ $\quad$ basis of $V\ (W)$

$n \quad m$ $\quad \dim(V\otimes W) = n \times m$

$\phi(g)(e_i) = e_k\, \phi_{ki} \qquad 1 \le i, k \le r$

$\psi(g)(E_j) = E_\ell\, \psi_{\ell j} \qquad 1 \le j, \ell \le r'$

$\chi_{\phi(g)} = \sum_{k=1}^{n} \phi_{kk}(g)$

$\chi_{\psi(g)} = \sum_{\ell=1}^{m} \psi_{\ell\ell}(g)$

$$\chi_{\phi\otimes\psi}(g) := \langle \underbrace{e_s^* \otimes E_t^*}, \ \phi\otimes\psi\,(e_s \otimes E_t) \rangle$$

$$= \langle e_s^* \otimes E_t^* \quad \phi_{ks}\, \psi_{\ell t} \quad e_k \otimes E_\ell \rangle$$

$$= \sum \phi_{kk}\, \psi_{\ell\ell} = \chi_\phi \cdot \chi_\psi .$$

The general fact: The maximum tori intersects every conjugacy class.

Namely $\forall x \in G \quad g x g^{-1} \in T$. $\quad \{gxg^{-1},\ g \in G\}$

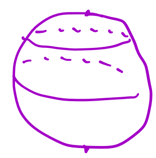
$x \in G$

This is a result of Cartan. $\chi(x) = \chi(gxg^{-1}) = \chi(t) \quad t \in T$

In this special case. We exam. the conjugacy class of

$$\boxed{e^{i\theta} = \underbrace{\cos\theta + i\sin\theta}}$$

$q = a + bj \qquad |q|^2 = 1$



$e^{i\theta}$

$$\underline{q\,(\underbrace{\cos\theta + i\sin\theta})\,\bar q} = \cos\theta + \boxed{q\,(i\sin\theta)\,\bar q}$$

$$= \cos\theta + \underline{\sin\theta}\quad q\,(i)\,\bar q$$

$$(a+bj)\,i\,(\bar a - j\bar b) = i|a|^2 - i|b|^2 + (-i\underbrace{b a})j + (\underline{i a b})j$$

$$= i\,(\underbrace{|a|^2 - |b|^2}) + \underbrace{(-2abi)j}$$

$i\,(c + dj)$

Hence

$$q\,(\cos\theta + i\sin\theta)\,\bar q = \cos\theta + \sin\theta \left[\, i(|a|^2 - |b|^2) - \underbrace{(2abi)j} \,\right]$$

$\theta \in \mathbb{R}$

$$(a, b) \in S^3 \longrightarrow (\underbrace{|a|^2 - |b|^2}, \underline{-2ab})$$

$(|a|^2 - |b|^2)^2 + 4|a|^2|b|^2$

$= 1$

$|a|^2 + |b|^2 = 1$ is a fibration of $S^3 \to S^2$

Hence the conjugacy class of $e^{i\theta}$ is a

2-sphere with radius $\sin\theta$ (the conjugacy class) except the case $\sin\theta = 0$

$$\phi_1(\underset{\in S^3}{e^{i\theta}}) = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$$

$V_k = \text{Span}$

$$\Rightarrow \begin{cases} \phi_1(z_1) = e^{i\theta} z_1 \\ \phi_1(z_2) = e^{-i\theta} z_2 \end{cases} \Rightarrow \quad \phi_k \text{ action on } \quad \{ \underbrace{z_1^k, z_1^{k-1} z_2, \dots, z_1 z_2^{k-1}, z_2^k} \}$$

$$\boxed{e^{ik\theta}}, \; e^{i(k-2)\theta}, \; \dots, \; e^{-i(k-2)\theta}, \; e^{-ik\theta}$$

$$\chi_{\phi_k}(e^{i\theta}) = \frac{e^{i(k+1)\theta} - e^{-i(k+1)\theta}}{e^{i\theta} - e^{-i\theta}} \qquad (1) \qquad \begin{bmatrix} e^{ik\theta} & & & \\ & e^{i(k-2)\theta} & & \\ & & \ddots & \\ & & & e^{-ik\theta} \end{bmatrix}$$

Note $|q \cdot x|^2 = q \cdot x \cdot \bar{x} \cdot \bar{q} = |x|^2 \Rightarrow R_q \; L_{\bar{q}}$ are isometry on S^3

$q \in S^3$

$\Rightarrow$ The $d\mu$ on S^3 is just the standard round metric volume $\cdot \boxed{\frac{1}{2\pi^2}}$

(since $|S^3| = \int_0^\pi 4\pi \sin^2\theta \, d\theta = 2\pi^2$).

$$\sin^2\theta = \frac{1}{4} \left| e^{i\theta} - e^{-i\theta} \right|^2 \qquad (2)$$

$$\Rightarrow \int_{S^3} |\chi_{\phi_k}|^2 \, d\mu = \int_0^\pi |\chi_{\phi_k}|^2 \cdot \frac{1}{2\pi^2} \, \boxed{4\pi \sin^2\theta} \, d\theta$$

$\sin^2\theta = \frac{1}{4}|e^{i\theta} - e^{-i\theta}|^2$

use (1) & (2)

$$= \frac{1}{\pi} \int_0^{2\pi} |\chi_{\phi_k}|^2 \, \frac{|e^{i\theta} - e^{-i\theta}|^2}{4} \, d\theta = 1.$$

$$\left(= \frac{1}{4\pi} \int_0^{2\pi} \left| e^{i(k+1)\theta} - e^{-i(k+1)\theta} \right|^2 d\theta = 1 \right)$$

Similarly $\int_{S^3} \chi_{\phi_k} \bar{\chi}_{\phi_l} = 0$ if $k \neq l$.

Now we know that $\phi_k : S^3 \to$ linear transformation induced on V_k

is irreducible

$\forall\, \psi : S^3 \longrightarrow V.$ we consider

$$\int_{S^3} \chi_\psi \cdot \overline{\chi_{\phi_\ell}}\, d\mu = \frac{1}{2\pi}\int_0^\pi \chi_\psi(e^{i\theta})\, \overline{\left(\frac{e^{i(\ell+1)\theta} - e^{-i(\ell+1)\theta}}{e^{i\theta}-e^{-i\theta}}\right)}\, |e^{i\theta}-e^{-i\theta}|^2\, d\theta$$

$$j\, e^{i\theta}\, \bar{j} = e^{-i\theta} \implies \chi_\psi(e^{i\theta}) = \chi_\psi(e^{-i\theta})$$

Hence $\Rightarrow$

$$\int_{S^3} \chi_\psi \cdot \overline{\chi_{\phi_\ell}}\, d\mu = \frac{1}{2\pi}\int_0^\pi \chi_\psi\,(e^{i\theta}-e^{-i\theta}) \cdot \overline{\left(e^{i(\ell+1)\theta}-e^{-i(\ell+1)\theta}\right)}$$

$2 \cdot \sin(\ell+1)\theta$

If $\langle \chi_\psi, \overline{\chi_{\phi_\ell}} \rangle = 0 \quad \forall\, \ell$

$\Rightarrow$ The odd function $\boxed{2\chi_\psi(e^{i\theta})\sin\theta}$ must be zero.

This shows $\phi_k : S^3 \longrightarrow GL(k+1, V_k)$ gives all irreducible representations of S^3.